Number Theory 15

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Q> $a \in \mathbb{Q}$. If $x = 11 + 11\sqrt{11a^2+1}$ is an odd integer then show that x is a perfect square.

$\gcd(n,d) = 1$

Ans:- $a = \frac{n}{d}$ $\quad 11\sqrt{11a^2+1}$ is even integer

$$\Rightarrow \frac{11}{d}\sqrt{11n^2+d^2} \text{ is even}$$

$2 \mid \left(\frac{11}{d}\sqrt{11n^2+d^2}\right)$ $\quad \frac{11}{d}\sqrt{11n^2+d^2} = 2k$

$$\Rightarrow 11^2(11n^2+d^2) = 4k^2d^2$$

$$= 11^3n^2 + 11^2d^2 = 4k^2d^2$$

Suppose $\exists$ Any prime p such that $p \mid d \Rightarrow p \mid 4k^2d^2 \Rightarrow p \mid (11^3n^2 + 11^2d^2)$

$$\Rightarrow p \mid 11^3 n^2$$

Now as $p \nmid n \Rightarrow p \mid 11^3 \Rightarrow p \mid 11 \Rightarrow p = 11$

As p is any prime $\Rightarrow d$ must be 11 if such a prime exists.

$\Rightarrow a = \frac{n}{11}$ (or just n if p doesn't exist)

$\Rightarrow 11\sqrt{11a^2+1} = \frac{11}{11}\sqrt{11n^2+11^2} = \sqrt{11(n^2+11)}$ must be even integer

$\Rightarrow \sqrt{11}\sqrt{n^2+11}$ must be even $\Rightarrow \sqrt{n^2+11} = \sqrt{11}\,(2k')$

$\Rightarrow n^2+11 = 11(2k')^2 \Rightarrow 11 \mid (n^2+11)$

$\Rightarrow 11 \mid n^2 \Rightarrow \gcd(d,n) \neq 1$

$\Rightarrow \Leftarrow$ So $d = 1$

$\Rightarrow a$ must be an integer

If a is even integer, say $2m$ then $\underbrace{11}_{\text{odd}}\underbrace{\sqrt{11(2m)^2+1}}_{\text{odd}}$ will be odd.

So a must be an odd integer.

$\Rightarrow a = 2m+1$ (let)

[If $n \in \mathbb{Z}$ $\sqrt{n}$ is a $\mathbb{Z}$ iff n is a perfect square]

$\Rightarrow 11\sqrt{11(2m+1)^2+1}$ must be even integer

$\Rightarrow 11(2m+1)^2+1 = 4k^2$ (say)

$11a^2+1 = s^2$

$\Rightarrow 11a^2 = s^2-1 \quad \Rightarrow 11a^2 = (s-1)(s+1) \qquad \gcd(s-1, s+1) = 1$

Prime $q \neq 11$ such that $q \mid (s+1) \Rightarrow q^{\alpha} \mid (s+1) \qquad s+1 = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q^{\alpha} \cdots q_n^{\alpha_n}$

(α must be even)

$x = 11 + 11\sqrt{11a^2+1} = 11 + 11s$

$= 11(s+1)$

$\Rightarrow 11(s+1)$ has all primes power as even except 11.

$\Rightarrow x = 11 \times 11^b c^2 \quad \ldots [b, c \in \mathbb{Z}]$

⊛ Rest of it can be done using Pell's equation for $\mathbb{Z}[\sqrt{11}]$ and peridiocity to show $s \equiv -1 \pmod{11}$.

Q) Show that $(a+b)^{p^i} \equiv a^{p^i} + b^{p^i} \pmod p$ if p is a prime and i is any non-negative integer

Ans:- $(a+b)^{p^i} = a^{p^i} + {}^{p^i}C_1 a^{p^i-1} b + \cdots + {}^{p^i}C_{p^i-1} a b^{p^i-1} + b^{p^i}$

$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_k a^{n-k} b^k + \cdots + {}^nC_n a^{n-n} b^n$

$= a^n + {}^nC_1 a^{n-1} b + \cdots + {}^nC_{n-1} a b^{n-1} + b^n$

$\Rightarrow (a+b)^{p^i} \equiv a^{p^i} + b^{p^i} \pmod p \quad \ldots \left(\text{as } \underbrace{p \mid {}^{p^i}C_k}_{\text{try to prove it also}} \ \forall\, k \geq 1 \ \&\ k \leq p^i-1\right)$

HomeWork:- Prove this question using induction on i.

HomeWork:- Prove that for prime p,

$$x^p - x \equiv x(x-1)(x-2)\cdots(x-(p-1)) \pmod p$$